International Lending with Moral Hazard and Risk of Repudiation
by

1 Model Setup

Infinite discrete time $t = 0, 1, \ldots, \infty$, and only one consumption good.

1.1 Two types of agents

One infinitely-lived risk averse agent: the borrower who has utility $u(c_t)$. $u' > 0$, $u'' < 0$, $u'(0) = +\infty$, and $u$ is bounded above by $\pi$. A sequence of OLG risk neutral agents: the lender born in time $t$ consumes $c^y_t$ when young and $c^o_{t+1}$ when old. Let $b_t = M - c^y_t$ denotes the loan and $d_{t+1} = c^o_{t+1} - M$ denotes the repayment. Note there are no constraints on the signs of these two variables.

1.2 Endowment

The borrower is endowed with $Y_0 - d_0$ at $t = 0$ and she can invest $I_t \in \mathbb{I}$ today to produce a random output $Y_{t+1}$ tomorrow, where $Y_{t+1} \in \mathbb{Y} \equiv \{Y_1, Y_2, \ldots, Y_N\}$ with $Y_i \geq Y_1 > 0$. The probability is $g_i(Y_{t+1} = Y_i|I_t = I) = g_i(I) > 0$ for all $i \in \{1, 2, \ldots, N\}$ and $I \in \mathbb{I}$. The lender has an endowment $M$ in each of the two periods she is alive.

1.3 Information

$I_t$ is unobservable to lender but the realization of $Y_t$ is observable to both parties and verifiable.

1.4 History

Entering any period $t+1$, the history is summarized by $Q^t = \{Q_0, Q_1, \ldots, Q_t\}$ where $Q_t = Y_t - d_t$.

1.5 Timing

see appendix.

2 Contract

The loan contract is denoted by the pair $(b_t, d_{t+1}(Y_{t+1}))$. An allocation $\sigma$ is defined to be a plan which specifies the deposition of the current output $Y_t$ between current consumption of the borrower, the old lender, the young lender, and investment. More specifically, an allocation includes the consumption of
the borrower, c, investment, I, loan, b, and repayment schedules, d, depending
on the history:

$$\sigma = \{c_t(Q_t), I_t(Q_t), b_t(Q_t), d_{t+1}(Q_t; Y_{t+1})\}_{t=0}^{\infty}.$$ 

**Definition 1** An allocation is feasible if for all $t \geq 0$, $Q_t, Y_t \in \mathbb{Y}$:

$$c_t(Q_t) - b_t(Q_t) + I_t(Q_t) \leq Y_t - d_t(Y_t)$$  \hspace{1cm} (1)

with $c_t(Q_t), b_t(Q_t) \geq 0, b_t, -d_t \leq M,$ and $Y_0, d_0, Q_0$ given.

Let $\sigma|_{Q_t}$ be the continuation of an allocation $\sigma$ conditional on the realization $Q_t$.

The value of an allocation $\sigma$ for the borrower is denoted by $U_B(\sigma)$ and characterized by:

$$U_B(\sigma) = (1-\delta)E_0\sum_{t=0}^{\infty} \delta^t u\left(c_t(Q_t)\right).$$

The lender born in period $t, t \geq 0$, values the continuation of an allocation $\sigma|_{Q_t}$ as follows:

$$U_L(t|_{Q_t}) = -b_t(Q_t) + \delta \sum_{Y_{t+1} \in \mathbb{Y}} g_t\left(I_t(Q_t)\right) d_{t+1}(Q_t; Y_{t+1}).$$

The reservation utility when the lenders in autarky is $0$ while the reservation utility that the borrower can obtain in autarky depends on her income, $Z$, at the time when there is no more borrowing in the future:

$$U_{B_{Autarky}}(Z) = \max_t (1-\delta)u(Z-I) + \delta \sum_{Y^{t+1} \in \mathbb{Y}} g_t\left(I_t\right) U_{B_{Autarky}}(Y^{t+1}).$$

**Assumption 1** $(1-\delta)u(0) + \bar{u} < U_{B_{Autarky}}(Y_1)$.

**Definition 2** An allocation is individually rational if for all $t \geq 0$, $Q_t$:

$$U_B(\sigma|_{Q_t}) \geq U_{B_{Autarky}}(Q_t) \quad \text{and} \quad U_L(t|_{Q_t}) \geq 0.$$  \hspace{1cm} (2)

**Definition 3** An allocation $\sigma$ is immune from the threat of repudiation if for all $t \geq 0$, $Q_t, Y_{t+1} \in \mathbb{Y}$, the continuation allocation $\sigma|_{Q^t; Y_{t+1}}$, after the realization of output $Y_{t+1}$ from investment in $t$, satisfies:

$$U_B(\sigma|_{Q^t; Y_{t+1}}) \geq U_{B_{Autarky}}(Y_{t+1}).$$  \hspace{1cm} (3)

**Definition 4** An allocation $\sigma \{c_t, I_t, b_t, d_{t+1}\}_{t=0}^{\infty}$ is incentive compatible if for all feasible allocations $\sigma' = \{c'_t, I'_t, b_t, d_{t+1}\}_{t=0}^{\infty}$ with the components $\{b_t, d_{t+1}\}_{t=0}^{\infty}$ unchanged:

$$U_B(\sigma) \geq U_B(\sigma').$$  \hspace{1cm} (4)
2.1 Constrained Pareto Optimal Contracting

\[
\max_{\sigma} U^B (\sigma) \tag{c}
\]

s.t. (1), (2), (3) and (4).

Next task is to restate the optimal contracting problem as a recursive problem. Remember in the repeated principal-agent problem, the state variable is the reservation utility promised to the agent. In the setting of international lending, $Q$, the amount of the consumption good the borrower has left after paying outstanding loans, is taken as the state variable. $Q \in \mathbb{Q}$, the set of all possible $Q$’s.

Define the borrower’s utility correspondence, $V$, to be the set of payoffs which the borrower can obtain from allocations which satisfy constraints (1)-(4) for each value of $Q \in \mathbb{Q}$. Define a set of current controls to be the vector $A = (c, I, b, d')$ where $c, I, \text{and } b$ are scalars and $d' : \mathbb{Y} \rightarrow \mathbb{R}$.

\[
V (Q) = \{ U^B (\sigma) | \sigma \text{ satisfies (1)-(4) and } Q_0 = Q \}.
\]

Note that $U^B_{\text{Autarky}} (Q) \in V (Q)$, and $V (Q)$ is thus not empty-valued. The value of the optimal contract as a function of the state variable $V^* (Q)$ is defined:

\[
V^* (Q) = \sup_{v \in V (Q)} v.
\]

Let $U (Q) : \mathbb{Q} \rightarrow \mathbb{R}$ be any continuation value function for the borrower given each value of $Q \in \mathbb{Q}$.

**FUNCTIONAL EQUATION 1 (p)**

\[
V^* (Q) = \sup_{A,U} (1-\delta)u(c) + \delta \sum_{Y' \in \mathbb{Y}} g_i (I) U (Y_i - d'(Y'_i)) \tag{p}
\]

Subject to:

\[
c + I - b \leq Q, \quad b, -d'(Y') \leq M, \quad c, I \geq 0, \quad (1')
\]

\[
(1-\delta)u(c)+\delta \sum_{Y' \in \mathbb{Y}} g_i (I) U (Y' - d'(Y')) \geq U^B_{\text{Autarky}} (Q) \quad \text{and } b \leq \delta \sum_{Y' \in \mathbb{Y}} g_i (I) d' (Y'), \quad (2')
\]

\[
U (Y' - d'(Y')) \geq U^B_{\text{Autarky}} (Y') \quad \forall Y' \in \mathbb{Y}, \quad (3')
\]

\[
I \in \arg \max_{\tilde{I} \in [0, Q+b]} (1-\delta)u(Q + b - \tilde{I}) + \delta \sum_{Y' \in \mathbb{Y}} g_i (\tilde{I}) U (Y' - d'(Y')) \tag{4'}
\]

**Proposition 5** Assume that the value function $V^*$ is continuous. Then the continuation value function $U^*$ which solves the program (p) necessarily satisfies $U^* = V^*$.
Proof. A sketch. ■
To be rigorous, what we did not show here are the existence of an optimal contract and the conditions needed for $V^*$ to be continuous. Prove that $V(Q)$ is a compact set, i.e., an optimal loan contract exists for each $Q$. Prove that under which conditions, $V^*(Q)$, which defines the value of the optimal contract for each initial value of the state variable, is continuous.

**FUNCTIONAL EQUATION 2 (p*)**

$$V^*(Q) = \max_A (1 - \delta)u(c) + \delta \sum_{Y' \in \mathcal{Y}} g_i(I)V^*(Y_i - d'(Y_i))$$  \hspace{1cm} (p*)

Subject to:

$$c + I - b \leq Q, \quad b, -d'(Y') \leq M, \quad c, I \geq 0,$$  \hspace{1cm} (1')

$$b \leq \delta \sum_{Y' \in \mathcal{Y}} g_i(I)d'(Y'),$$  \hspace{1cm} (2')

$$V^*(Y' - d'(Y')) \geq U^*_{\text{Autarky}}(Y') \quad \forall Y' \in \mathcal{Y},$$  \hspace{1cm} (3')

$$I \in \arg \max_{I \in [0, Q + b]} (1 - \delta)u(Q + b - I) + \delta \sum_{Y' \in \mathcal{Y}} g_i(I)V^*(Y' - d'(Y')).$$  \hspace{1cm} (4')

### 2.2 The Optimal Pattern of Capital Flows

**Assumption 2** Assume that the distribution of output given investment $g_i(I)$ is given by the convex combination of two underlying distributions $g_i^b$ and $g_i^l$ as follows:

$$g_i(I) = \lambda(I)g_i^b + (1 - \lambda(I))g_i^l$$

with $g_i^b / g_i^l$ monotone in $i, \lambda(I) \in [0, 1], \lambda'(I) > 0,$ and $\lambda'(I) \leq 0$ for all $I$.

**Assumption 3** Assume that the value of repayments at the optimum is increasing in investment:

$$\sum_i (g_i^b - g_i^l)d'(Y'_i) \geq 0.$$  

This is really saying that, at the optimum, the lender would prefer that the borrower make larger rather than smaller investments.

**Assumption 4** Assume that the constrained optimal investment level is interior.

The assumptions above imply that the optimal incentive compatible level of investment is the unique solution to the first order condition:

$$-(1 - \delta)u'(Q + b - I) + \delta \lambda'(I) \sum_i (g_i^b - g_i^l)V^*(Y'_i - d'(Y'_i)) = 0.$$
FUNCTIONAL EQUATION 3 \((p_L)\)

\[
\sup_{A,U_d} (1 - \delta)u(c) + \delta \sum_i g_i(I) U_d(Y_i')
\]

Subject to:

\[
\begin{align*}
& c + I - b \leq Q, \quad (1') \\
& b \leq \delta \sum_i g_i(I) d'(Y_i'), \quad (2') \\
& U_d(Y_i') \geq U^{\text{Autarky}}_d(Y_i'), \quad \forall Y_i', \quad (3') \\
& -(1 - \delta)u'(Q + b - I) + \delta \sum_i g_i(I) U_d(Y_i') \geq 0, \quad (4') \\
& V^*(Y_i' - d'(Y_i')) \geq U_d(Y_i'), \quad (5')
\end{align*}
\]

where \(U_d = (U(Y_1' - d'(Y_1')), ..., U(Y_N' - d'(Y_N')) = (U_d(Y_1'), ..., U_d(Y_N'))\) and \(g_i(I) = \lambda(I) (g_i^b - g_i^s)\).

The Lagrangian for program \((p_L)\) is written as the following:

\[
L(A, U_d, \mu) = (1 - \delta)u(c) + \delta \sum_i g_i(I) U_d(Y_i') + \mu_1 (Q + b - c - I) + \mu_2 \left( \delta \sum_i g_i(I) d'(Y_i') - b \right) + \mu_3 \left( \sum_i \mu_3 g_i(I) \left( U_d(Y_i') - U^{\text{Autarky}}_d(Y_i') \right) \right) + \mu_4 \left( -(1 - \delta)u'(Q + b - I) + \delta \sum_i g_i(I) U_d(Y_i') \right) + \mu_5 \left( \sum_i \mu_5 g_i(I) \left( V^*(Y_i' - d'(Y_i')) - U_d(Y_i') \right) \right).
\]

The first order condition with respect to \(U_d(Y_i')\) from the Lagrangian \(L\) is written:

\[
1 + \mu_4 \left( \frac{g_i(I)}{g_i(I)} \right) = \mu_5(Y_i') - \mu_3(Y_i').
\]

The conclusion is thus, when constraint \((3')\) binds, the borrower experiences a capital outflow. This will explain the most striking feature of international lending: foreign lenders demand capital exports from borrowers who have suffered adverse shocks and cause these borrowers to suffer a fall in consumption and investment.

Appendix
Young Lender $t+1$ is born

The output is realized $Y_{t+1}$

Borrower $b_{t+1}$ loans from the young

Consume

Payback $d_{t+1}(Y_{t+1})$ to the old lender born in $t$

Borrower makes investment $I_{t+1}$

$Q_t$ to $Q_{t+1}$